Semester Exam Ergodic Theory Time: 3hrs Total Marks: 50
Answer any five and each question is worth 10 Marks Total Marks 50

1. Let $T\colon X \to X$ be a measure preserving transformation of the probability space $(X, \mathcal{B}, m)$. Prove that T is ergodic on X if and only if for any $B \in \mathcal{B}$ with $m(T^{-1}B \triangle B) = 0$, we have $m(B) = 0$ or 1 if and only if for any $f \in L^2(X)$, $f(T(x)) = f(x)$ a.e, we have f is constant a.e.

2. (a) Define one-sided $(p_0, p_1, \cdots, p_{k-1})$-shift and prove that it is measure preserving but not invertible and is ergodic.

   (b) Let $T\colon X \to X$ be a measure-preserving transformation of a probability space $(X, \mathcal{B}, m)$. Show that T is strong mixing if and only if $T \times T$ is strong mixing.

3. Let $T\colon X \to X$ be a measure-preserving transformation of a probability space $(X, \mathcal{B}, m)$ and $A \subset X$ be a set of positive measure. Define $T_A\colon A \to A$ by $T_A(x) = T^{n_A(x)}(x)$ for a.e. $x \in A$ where $n_A(x)$ is the least positive integer such that $T^{n_A(x)}(x) \in A$. Prove that T_A is a measure preserving transformation on (A, m_A) with $m_A(E) = \frac{m(E)}{m(A)}$ for any measurable set E in A and if T is ergodic show that T_A is ergodic.

4. Let $(X, \mathcal{B}, m)$ be a probability space. Let $\mathcal{A}, \mathcal{C}$ and $\mathcal{D}$ be finite subσ-algebras. Then prove that

   (a) $H(\mathcal{A} \bigvee \mathcal{C}/\mathcal{D}) = H(\mathcal{A}/\mathcal{D}) + H(\mathcal{C}/\mathcal{A} \bigvee \mathcal{D})$ and

   (b) $\mathcal{C} \subset \mathcal{D}$ implies $H(\mathcal{A}/\mathcal{C}) \geq H(\mathcal{A}/\mathcal{D})$.

5. Let $(X, \mathcal{B}, m)$ be a probability space. Let $\mathcal{A}$ and $\mathcal{C}$ be a finite subσ-algebras of $\mathcal{B}$. Prove that

   (a) $H(\mathcal{A}/\mathcal{C}) = 0$ if and only if $\mathcal{A} \subset^0 \mathcal{C}$;

   (b) $H(\mathcal{A}/\mathcal{C}) = H(\mathcal{A})$ if and only if $\mathcal{A}$ and $\mathcal{C}$ are independent.

6. (a) If T_1 and T_2 are measure preserving systems such that T_2 is a factor of T_1, then show that $h(T_2) \leq h(T_1)$.

   (b) Prove that entropy of a measure preserving transformation is conjugacy invariant

7. (a) Prove that measure preserving transformation of a finite space has zero entropy.

   (b) Prove that the two-sided (p, P) Markov shift has entropy $-\sum_{i,j} p_i p_{ij} \log p_{ij}$.